Sets

DEFINITION: A **set** is an unordered collection of objects (**elements**)

- A set **contains** its elements; $a \in A$ ($a \notin A$): a is (not) an element of A.
- How to describe a set:
  - **roster method**: enumerate the members of a set
    - $\left\{7, \left\{8, \{9\}\right\}\right\}$
  - **set builder**: $A = \{a : P(a)\}$ (or $A = \{a | P(a)\}$)
    - $A = \{a : a^2 - 3a + 2 = 0\}$
- **empty set**: The set that has no elements (null set, ∅)
- **universal set**: the set containing all objects under consideration
- $A = B$: $\forall x \, (x \in A \leftrightarrow x \in B)$ --- A is **equal** to B
- $A \subseteq B$: $\forall \, x \, (x \in A \rightarrow x \in B)$ ---A is a **subset** of B
- $A \subset B$: $(A \subseteq B) \wedge (A \neq B)$ --- A is a **proper subset** of B

Set Operations

DEFINITION: Let $A, A_1, \ldots, A_n, B, C$ be any sets. Let S be a universal set.

- $A \cup B = \{x | x \in A \vee \ x \in B\}$: the **union** of A and B
- $A \cap B = \{x | x \in A \wedge x \in B\}$: the **intersection** of A and B
  - **disjoint**: $A \cap B = \emptyset$
- $A - B = \{x | x \in A \wedge x \notin B\}$: the **difference** of A and B
  - the **complement** of B with respect to A
- $A \bigoplus B = (A - B) \cup (B - A)$: the **symmetric difference** of A and B
- $\mathcal{P}(A) = \{x | x \subseteq A\}$: the **power set** of A
- $A \times B = \{(a, b) : a \in A \wedge b \in B\}$: the **Cartesian product** of A and B
  - $A_1 \times \cdots \times A_n = \{(a_1, \ldots, a_n) : a_1 \in A_1, \ldots, a_n \in A_n\}$

REMARK: Venn Diagrams—represent set operations

Set Operations

Generalized union: $\cup A = \{x | \exists y(y \in A \wedge x \in y)\}$--$A$ is a set of sets.

- $\cup \{A_1, A_2, \dots, A_n\} = \bigcup_{i=1}^{n} A_i$
  - Let $A_i = \{i, i+1, \dots\}$. Then $\bigcup_{i=1}^{n} A_i = ?$
- $\cup \{A_1, A_2, \dots\} = \bigcup_{i=1}^{\infty} A_i$
  - Let $A_i = \{1,2, \dots, i\}$. Then $\bigcup_{i=1}^{\infty} A_i = ?$
- $\cup \{A_i : i \in I\} = \cup_{i \in I} A_i$
  - Let $A_i = (i - 1/2, i + 1/2)$ for all $i \in [0,1]$. Then $\cup_{i \in I} A_i = ?$
- $\cup \emptyset = \emptyset$

Generalized intersection: $\cap A = \{x | \forall y(y \in A \rightarrow x \in y)\}$

- $\cap \{A_1, A_2, \dots, A_n\} = \bigcap_{i=1}^{n} A_i$
- $\cap \{A_1, A_2, \dots\} = \bigcap_{i=1}^{\infty} A_i$
- $\cap \{A_i : i \in I\} = \bigcap_{i \in I} A_i$
- $\cap \emptyset$ is not meaningful

Laws of Set Operations

Complementation law(双补律): $\bar{\bar{A}} = A$

Commutative Laws(交换律):

- $A \cup B = B \cup A$
- $A \cap B = B \cap A$

Associative Laws(结合律):

- $A \cup (B \cup C) = (A \cup B) \cup C$
- $A \cap (B \cap C) = (A \cap B) \cap C$

Absorption Laws(吸收律):

- $A \cup (A \cap B) = A$
- $A \cap (A \cup B) = A$

De Morgan's Laws (摩根律):

- $\overline{A \cap B} = \bar{A} \cup \bar{B}$
- $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Identity Laws (同一律):

- $A \cap S = A$
- $A \cup \emptyset = A$

Idempotent Laws(等幂律):

- $A \cup A = A$;
- $A \cap A = A$

Domination Laws(零律):

- $A \cup S = S$
- $A \cap \emptyset = \emptyset$

Negation Laws(补余律):

- $A \cup \bar{A} = S$
- $A \cap \bar{A} = \emptyset$

- $\bar{S} = \emptyset$
- $\bar{\emptyset} = S$

Distributive Laws(分配律):

- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

S is the universal set

Function

DEFINITION: Let $A, B \neq \emptyset$ be two sets.

- A **function (map)** $f: A \to B$ assigns a unique element $b \in B$ for every $a \in A$.
  - $f(a) = b$: b is the **image**象 of a and a is a **preimage**原象 of b.
  - A is the **domain**定义域 of f and B is the **codomain**陪域 of f.
- The **image** of $X \subseteq A$ is $f(X) = \{f(x) | x \in X\}$. The **preimage** of $Y \subseteq B$ is $f^{-1}(Y) = \{x: x \in A \land f(x) \in Y\}$. The **range**值域 of f is $f(A)$
- $f: A \to B$ is said to be
  - **injective**单射 if "$f(a) = f(b) \Rightarrow a = b$"
  - **surjective**满射 if $f(A) = B$
  - **bijective**双射 if it is injective and surjective.
- $g: A \to B$ and $f: B \to C$. $f \circ g(x) = f(g(x))$ is the **composition**复合 of f, g.
- If $f: A \to B$ is a bijection, then f has an **inverse function**反函数 $f^{-1}: B \to A$.

Cardinality of Sets

DEFINITION: Let A be a set.

- A is called a **finite set**有限集 if it has finitely many elements.
  - The **cardinality**基数 $|A|$ of a finite set A is the number of elements in A.
- A is called an **infinite set**无限集 if it is not a finite set.

EXAMPLE: $\emptyset, \{1\}, \{x: x^2 - 2x - 3 = 0\}, \{a, b, c, \dots, z\}$ are all finite sets

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ are all infinite sets

DEFINITION: Let A, B be any sets.

- A, B **have the same cardinality**等势 ($|A| = |B|$) if there is a bijection $f: A \to B$
- We say that $|A| \leq |B|$ if there exists an injection $f: A \to B$.
  - If $|A| \leq |B|$ and $|A| \neq |B|$, we say that $|A| < |B|$

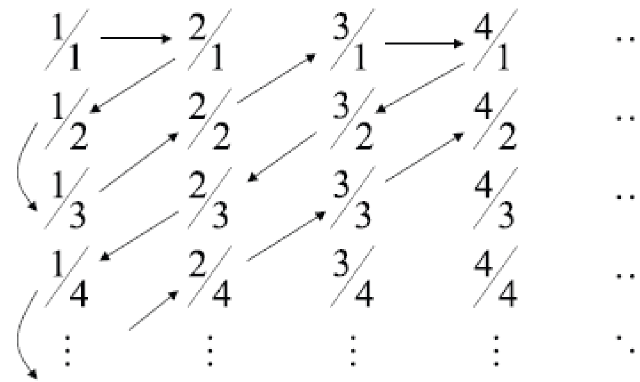
THEOREM: Let A, B, C be any sets. Then

- $|A| = |A|$
- $|A| = |B| \Rightarrow |B| = |A|$
- $|A| = |B| \wedge |B| = |C| \Rightarrow |A| = |C|$

Cardinality of Sets

EXAMPLE: $|\mathbb{Z}^+| = |\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}^+| = |\mathbb{Q}|$

- $f\colon \mathbb{Z}^+ \to \mathbb{N} \quad x \mapsto x - 1$
- $f\colon \mathbb{Z} \to \mathbb{N} \quad f(x) = \begin{cases} 2x & x \geq 0 \\ -(2z+1) & x < 0 \end{cases}$



$f\colon \mathbb{Z}^+ \to \mathbb{Q}^+$

EXAMPLE: $|\mathbb{R}^+| = |\mathbb{R}| = |(0,1)| = |[0,1]|$

- $f\colon \mathbb{R} \to \mathbb{R}^+ \quad x \mapsto 2^x$
- $f\colon (0,1) \to \mathbb{R} \quad x \mapsto \tan\big(\pi(x - 1/2)\big)$
- $f\colon [0,1] \to (0,1)$
  - $f(1) = 2^{-1}, f(0) = 2^{-2}, f(2^{-n}) = 2^{-n-2}, n = 1,2,3, \ldots$
  - $f(x) = x$ for all other x

EXAMPLE: $|2^X| = |\mathcal{P}(X)|$

- $2^X = \{\, \alpha |\, \alpha\colon X \to \{0,1\}$ is a function$\}$ the set of all functions from X to $\{0,1\}$
- $\mathcal{P}(X) = \{A | A \subseteq X\}$: the power set of X
- $f\colon 2^X \to \mathcal{P}(X) \quad \alpha \mapsto A = \{x\colon \alpha(x) = 1\}$

Cardinality of Sets

THEOREM: $|(0,1)| \neq |\mathbb{Z}^+|$

- Suppose that $|(0,1)| = |\mathbb{Z}^+|$. Then there is a bijection $f: \mathbb{Z}^+ \rightarrow (0,1)$

$$f(1) = 0.b_{11}b_{12}b_{13}b_{14}b_{15}b_{16}b_{17}b_{18}b_{19}\cdots$$
$$f(2) = 0.b_{21}b_{22}b_{23}b_{24}b_{25}b_{26}b_{27}b_{28}b_{29}\cdots$$
$$f(3) = 0.b_{31}b_{32}b_{33}b_{34}b_{35}b_{36}b_{37}b_{38}b_{39}\cdots$$
$$f(4) = 0.b_{41}b_{42}b_{43}b_{44}b_{45}b_{46}b_{47}b_{48}b_{49}\cdots$$
$$f(5) = 0.b_{51}b_{52}b_{53}b_{54}b_{55}b_{56}b_{57}b_{58}b_{59}\cdots$$
$$f(6) = 0.b_{61}b_{62}b_{63}b_{64}b_{65}b_{66}b_{67}b_{68}b_{69}\cdots$$
$$\cdots$$
$$f(n) = 0.b_{n1}b_{n2}b_{n3}b_{n4}b_{n5}b_{n6}b_{n7}b_{n8}b_{n9}\cdots$$
$$\cdots$$

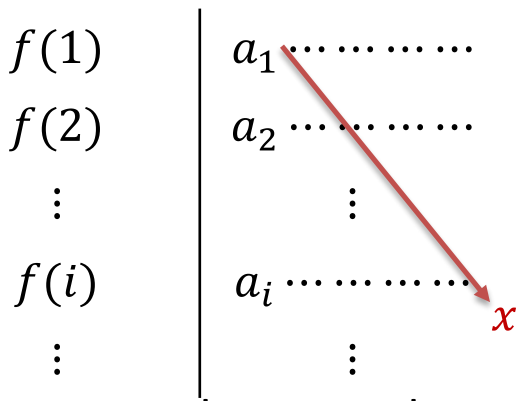
- Let $b_i = \begin{cases} 4, & b_{ii} \neq 4 \\ 5, & b_{ii} = 4 \end{cases}$ for $i = 1,2,3,\ldots$
- $b = 0.b_1b_2b_3b_4b_5b_6b_7b_8b_9\cdots$ is in $(0,1)$ but has no preimage
  - $b \neq f(i)$ for every $i = 1,2,\ldots$
- f cannot be a bijection

Cantor's Diagonal Argument

Question: Show that $|A| \neq |\mathbb{Z}^+|$.

The Diagonal Argument:

1) Suppose that $|A| = |\mathbb{Z}^+|$. Then there is a bijection $f\colon \mathbb{Z}^+ \to A$
2) Represent the function f as a list:



- Every element of $\mathbb{Z}^+$ appears once in the left-hand side
- Every element of A appears once in the right-hand side

3) Construct an element x by considering the diagonal of the list
4) Show that $x \neq a_i$ for all $i \in \mathbb{Z}^+$
5) Show that $x \in A$
6) 4) and 5) give a contradiction

Cantor's Theorem

THEOREM: (Cantor) Let A be any set. Then $|A| < |\mathcal{P}(A)|$.

- $|A| \leq |\mathcal{P}(A)|$
  - The function $f: A \rightarrow \mathcal{P}(A)$ defined by $f(a) = \{a\}$ is injective.
- $|A| \neq |\mathcal{P}(A)|$
  - Assume that there is a bijection $g: A \rightarrow \mathcal{P}(A)$
  - Define $X = \{a: a \in A \text{ and } a \notin g(a)\}$
  - ***X* should appear in the list**. It is clear that $X \subseteq A$ and hence $X \in \mathcal{P}(A)$
  - ***X* will not appear in the list**. Suppose that $X = g(x)$ for some $x \in A$
    - If $x \in X$, then $x \notin g(x) = X$
      - This gives a contradiction
    - If $x \notin X$, then $x \in g(x) = X$
      - This gives a contradiction

The Halting Problem

$$\mathbf{HALT}(P, I) = \begin{cases} \text{"halts"} & \text{if } P(I) \text{ halts;} \\ \text{"loops forever"} & \text{if } P(I) \text{ loops forever.} \end{cases}$$

- P: a program; I: an input to the program P.

QUESTION: Is there a Turing machine $\mathbf{HALT}$?

- Turing machine: can be represented as a an element of $\{0,1\}^*$
  - $\{0,1\}^* = \cup_{n\geq 0}\{0,1\}^n$: the set of all finite bit strings

THEOREM: There is no Turing machine $\mathbf{HALT}$.

- Assume there is a Turing machine $\mathbf{HALT}$
- Define a new Turing machine $\mathbf{Turing}(P)$ that runs on any Turing machine P
  - **If** $\mathbf{HALT}(P, P) =$ "halts“, loops forever
  - **If** $\mathbf{HALT}(P, P) =$ "loops forever“, halts
- $\mathbf{Turing}(\mathbf{Turing})$ loops forever$\Rightarrow$ $\mathbf{HALT}(\mathbf{Turing}, \mathbf{Turing}) =$ "halts"$\Rightarrow$**Turing(Turing) halts**
- $\mathbf{Turing}(\mathbf{Turing})$ halts $\Rightarrow$ $\mathbf{HALT}(\mathbf{Turing}, \mathbf{Turing}) =$ "loops forever"$\Rightarrow$**Turing(Turing)** loops forever