Countable and Uncountable

DEFINITION: A set A is **countable**可数，可列 if $|A| < \infty$ or $|A| = |\mathbb{Z}^+|$; otherwise, it is said to be **uncountable**不可数，不可列.

- countably infinite: $|A| = |\mathbb{Z}^+|$

EXAMPLE:

- $\mathbb{Z}^-, \mathbb{Z}^+, \mathbb{Z}, \mathbb{Q}^-, \mathbb{Q}^+, \mathbb{Q}, \mathbb{N}, \mathbb{N} \times \mathbb{N}$, are countable
- $\mathbb{R}^-, \mathbb{R}^+, \mathbb{R}, (0,1), [0,1], (0,1], [0,1), (a, b), [a, b]$ are uncountable

THEOREM: A set A is countably infinite iff its elements can be arranged as a sequence $a_1, a_2, \ldots$

- If A is countably infinite, then there is a bijection $f: \mathbb{Z}^+ \to A$
  - $a_i = f(i)$ for every $i = 1,2,3 \ldots$
- If $A = \{a_1, a_2, \ldots\}$, then the function $f: \mathbb{Z}^+ \to A$ defined by $f(i) = a_i$ is a bijection

Countable and Uncountable

THEOREM: Let A be countably infinite, then any infinite subset $X \subseteq A$ is countable.

- Let $A = \{a_1, a_2, \dots\}$. Then $X = \{a_{i_1}, a_{i_2}, \dots\}$
  - X is countable

THEOREM: Let A be uncountable, then any set $X \supseteq A$ is uncountable.

- If X is countable, then A is finite or countably infinite

THEOREM: If A, B are countably infinite, then so is $A \cup B$

- $A = \{a_1, a_2, a_3, \dots\}, B = \{b_1, b_2, b_3, \dots\}$
- $A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, \dots\}$ //no elements will be included twice
  - application: the set of irrational numbers is uncountable

THEOREM: If A, B are countably infinite, then so is $A \times B$

- $A = \{a_1, a_2, a_3, \dots\}, B = \{b_1, b_2, b_3, \dots\}$
- $A \times B = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_1, b_3), (a_2, b_2), (a_3, b_1), (a_1, b_4), \dots\}$

Schröder-Bernstein Theorem

QUESTION: How to compare the cardinality of sets in general?

- $|\mathbb{Z}^-| = |\mathbb{Z}^+| = |\mathbb{Z}| = |\mathbb{Q}^-| = |\mathbb{Q}^+| = |\mathbb{Q}| = |\mathbb{N}| = |\mathbb{N} \times \mathbb{N}|$
- $|\mathbb{R}^-| = |\mathbb{R}^+| = |\mathbb{R}| = |(0,1)| = |[0,1]| = |(0,1]| = |[0,1)|$
- $|\mathbb{Z}^+| \neq |(0,1)|$: hence, $|\mathbb{Z}^+| \neq |\mathbb{R}|$, and in fact $|\mathbb{Z}^+| < |\mathbb{R}|$
- $|\mathbb{Z}^+| < |\mathcal{P}(\mathbb{Z}^+)|$
- $|\mathbb{R}|?\ |\mathcal{P}(\mathbb{Z}^+)|$: which set has more elements?

THEOREM: If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

EXAMPLE: Show that $|(0,1)| = |[0,1)|$

- $|(0,1)| \leq |[0,1)|$
  - $f: (0,1) \to [0,1)\ \ x \to \frac{x}{2}$ is injective
- $|[0,1)| \leq |(0,1)|$
  - $g: [0,1) \to (0,1)\ \ x \to \frac{x}{4} + \frac{1}{2}$ is injective

Schröder-Bernstein Theorem

EXAMPLE: $|\mathcal{P}(\mathbb{Z}^+)| = |[0,1)| = (|\mathbb{R}|)$

- $|\mathcal{P}(\mathbb{Z}^+)| \leq |[0,1)|$
  - $f: \mathcal{P}(\mathbb{Z}^+) \to [0,1) \quad \{a_1, a_2, \dots\} \mapsto 0.\cdots 1_{a_1} \cdots 1_{a_2} \cdots$ is an injection.
- $|[0,1)| \leq |\mathcal{P}(\mathbb{Z}^+)|$
  - $\forall x \in [0,1), x = 0.r_1 r_2 \cdots \ \left(r_1, r_2, \cdots \in \{0, \dots, 9\}, \text{no } \dot{9}\right)$
    - $0 \leftrightarrow 0000, 1 \leftrightarrow 0001, \dots, 9 \leftrightarrow 1001$
    - x has a binary representation $x = 0.b_1 b_2 \cdots$
      - $f: [0,1) \to \mathcal{P}(\mathbb{Z}^+) \ \ x \mapsto \{i : i \in \mathbb{Z}^+ \wedge b_i = 1\}$ is an injection

THEOREM: $|\mathbb{Z}^+| < |\mathcal{P}(\mathbb{Z}^+)| = |[0,1)| = |(0,1)| = |\mathbb{R}|$

$\aleph_0$ $\quad 2^{\aleph_0}$ $\quad c$

The continuum hypothesis连续统假设**:** There is no cardinal number between $\aleph_0$ and c, i.e., there is no set A such that $\aleph_0 < |A| < c$.

Combinatorics

Enumerative combinatorics

- permutations, combinations, partitions of integers, generating functions, combinatorial identities, inequalities ……

Designs and configurations

- block designs, triple systems, Latin squares, orthogonal arrays, configurations, packing, covering, tiling ……

Graph theory

- graphs, trees, planarity, coloring, paths, cycles, ……

Extremal combinatorics

- extremal set theory, probabilistic method……

Algebraic combinatorics

- symmetric functions, group, algebra, representation, group actions……

Parenthesization

PROBLEM: Let $a_1, a_2, \dots, a_n, a_{n+1}$ be $n+1$ numbers. Let $*$ be any binary operator. Let C_n be the number of different ways of parenthesizing

$$a_1 * a_2 * \cdots * a_n * a_{n+1}$$

such that the calculation is not ambiguous. What is C_n?

- $n = 4$: there are 5 different ways
  - $((a_1 * a_2) * a_3) * a_4$
  - $(a_1 * a_2) * (a_3 * a_4)$
  - $(a_1 * (a_2 * a_3)) * a_4$
  - $a_1 * ((a_2 * a_3) * a_4)$
  - $a_1 * (a_2 * (a_3 * a_4))$
- $n = 100$?

Combinatorial Counting Techniques Required

Basic Rules of Counting

DEFINITION: Let A be a finite set. A **partition**划分 of set A is a family $\{A_1, A_2, \ldots, A_k\}$ of nonempty subsets of A such that

- $\bigcup_{i=1}^{k} A_i = A$ and
- $A_i \cap A_j = \emptyset$ for all $i, j \in [k]$ with $i \neq j$.

The Sum Rule加法原则: Let A be a finite set. Let $\{A_1, A_2, \ldots, A_k\}$ be a partition of A. Then $|A| = |A_1| + |A_2| + \cdots + |A_k|$.

- Suppose that a task can be done in one of n_1 ways, in one of n_2 ways, . . . , or in one of n_k ways, where none of the set of n_i ways of doing the task is the same as any of the set of n_j ways, for all pairs i and j with $1 \leq i < j \leq k$. Then the number of ways to do the task is $n_1 + n_2 + \cdots + n_k$.

Basic Rules of Counting

The Product Rule乘法原则: Let $A_1, A_2, \ldots, A_k$ be finite sets. Then

$$|A_1 \times A_2 \times \cdots \times A_k| = |A_1| \times |A_2| \times \cdots \times |A_k|. \ (*)$$

- Suppose that a procedure is carried out by performing the tasks $T_1, T_2, \ldots, T_k$ in sequence. If each task T_i $(i = 1,2, \ldots, k)$ can be done in n_i ways, regardless of how the previous tasks were done, then there are $n_1 n_2 \cdots n_k$ ways to carry out the procedure.

EXAMPLE: # of composite divisors of $N = 2^{100} \times 3^{200} \times 5^{1000}$.

- $A = \{n \in \mathbb{Z}^+ : n | N\}$; $|A| = 101 \times 201 \times 1001$ //product rule $n = 2^a 3^b 5^c$
- $A_1 = \{n \in A: n \text{ is prime}\}$; $A_2 = \{n \in A: n \text{ is composite}\}$; $A_3 = \{1\}$
  - $\{A_1, A_2, A_3\}$ is a partition of A.
  - $|A| = |A_1| + |A_2| + |A_3| \Rightarrow |A_2| = |A| - |A_1| - |A_3|$
  - $|A_1| = 3, |A_3| = 1; |A_2| = 101 \times 201 \times 1001 - 3 - 1 = 20321297.$

The Bijection Rule一一对应原则、相等原则: Let A and B be two finite sets. If there is a bijection $f: A \to B$, then $|A| = |B|$.

Permutations of Set

DEFINITION: Let A be a finite set of n elements. Let $r \in [n]$.

- $\boldsymbol{r}$**-permutation**$_{r\text{-排列}}$ of A: a sequence $a_1, a_2, \cdots, a_r$ of r distinct elements of A.
  - An n-permutation of A is simply called a **permutation**全排列 of A
    - Example: $A = \{1,2,3\}$
      - 2-Permutations of A: 1,2; 1,3; 2,1; 2,3; 3,1; 3,2
  - $P(n, r)$: the number of different r-permutations of an n-element set

THEOREM: $P(n, r) = n!/(n-r)!$ for all $n \in \mathbb{Z}^+$ and $r \in [n]$.

DEFINITION: Let A be a finite set of n elements.

- $\boldsymbol{r}$**-permutation** of A **with repetition**: a sequence $a_1, a_2, \cdots, a_r$ of r elements of A.
  - Example: $A = \{1,2,3\}$
  - 2-Permutations of A with repetition: 1,1; 1,2; 1,3; 2,1; 2,2; 2,3; 3,1; 3,2; 3,3

THEOREM: An n-element set has n^r different r-permutations with repetition.

Multiset

DEFINITION: A **multiset**多重集 is a collection of elements which are not necessarily different from each other.

- An element $x \in A$ has **multiplicity**重数 m if it appears m times in the multiset A.
- A multiset A is called an $\boldsymbol{n}$**-multiset**n-多重集 if it has n elements.
- $A = \{n_1 \cdot a_1, n_2 \cdot a_2, \ldots, n_k \cdot a_k\}$: an $(n_1 + n_2 + \cdots + n_k)$-multiset where the elements $a_1, a_2, \ldots, a_k$ has multiplicities $n_1, n_2, \ldots, n_k$, respectively.
- $T = \{t_1 \cdot a_1, t_2 \cdot a_2, \ldots, t_k \cdot a_k\}$ is called an $\boldsymbol{r}$**-subset** of A if
  - $0 \leq t_i \leq n_i$ for every $i \in [k]$, and
  - $t_1 + t_2 + \cdots + t_k = r$

EXAMPLE: $A = \{1 \cdot \mathrm{a}, 2 \cdot \mathrm{b}, 3 \cdot \mathrm{c}, 100 \cdot \mathrm{z}\}, T = \{1 \cdot \mathrm{b}, 98 \cdot \mathrm{z}\}$

- A is a 106-multiset; the multiplicities of a, b, c, z are 1,2,3,100, respectively
- T is a 99-subset of A

Permutations of Multiset

DEFINITION: Let $A = \{n_1 \cdot a_1, n_2 \cdot a_2, \dots, n_k \cdot a_k\}$ be an n-multiset.

- **permutation of** A: a sequence $x_1, x_2, \cdots, x_n$ of n elements, where a_i appears exactly n_i times for every $i \in [k]$.
- r**-permutation of** A: a permutation of some r-subset of A
  - $A = \{1 \cdot \mathrm{a}, 2 \cdot \mathrm{b}, 3 \cdot \mathrm{c}\}$
  - a, b, c, b, c, c is a permutation of A; bcb is a 3-permutation of A;
  - bcb is a permutation of the subset $\{2 \cdot \mathrm{b}, 1 \cdot \mathrm{c}\}$

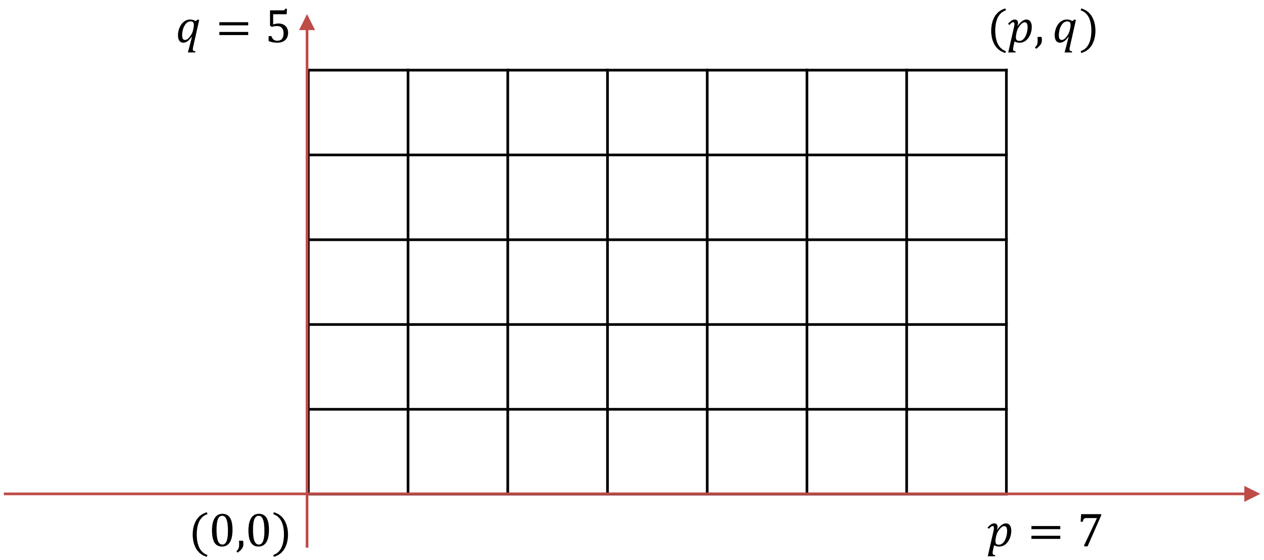
REMARK: Let $A = \{a_1, a_2, \dots, a_n\}$ be a set of n elements.

- For every $r \in [n]$, an r-permutation of A without repetition is an r-permutation of $\{1 \cdot a_1, 1 \cdot a_2, \dots, 1 \cdot a_n\}$.
- For every $r \geq 1$, an r−permutation of A with repetition is an r−permutation of $\{\infty \cdot a_1, \infty \cdot a_2, \dots, \infty \cdot a_n\}$.

THEOREM: Let $A = \{n_1 \cdot a_1, n_2 \cdot a_2, \dots, n_k \cdot a_k\}$ be a multiset. Then A has exactly $\frac{(n_1+n_2+\cdots+n_k)!}{n_1!n_2!\cdots n_k!}$ permutations.

Shortest Path

DEFINITION: A $\boldsymbol{p} \times \boldsymbol{q}$**-grid** is a collection of pq squares of side length 1, organized as a rectangle of side length p and q.



THEOREM: The number of shortest paths from $(0,0)$ to (p, q) is $\frac{(p+q)!}{p!q!}$.

- Let $A = \{p \cdot \rightarrow, q \cdot \uparrow\}$ be a $(p + q)$-multiset.
- # of shortest paths=# of permutations of A.